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SPECIFIC FEATURES OF SOLVING THE PROBLEM OF COMPRESSION OF AN ORTHOTROPIC PLASTIC MATERIAL BETWEEN ROTATING PLATES

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An instantaneous flow with compression of a wedge-shaped layer of a rigid-plastic orthotropic material between rotating plates is considered under the assumption that the principal axes of anisotropy are rays emanating from the wedge angle and lines orthogonal to them and that the maximum friction law is valid on the plate surfaces. The solution is reduced to quadratures, and its asymptotic analysis is performed. It is found that the solution is singular near the friction surface in the general case, and conditions at which the singularity disappears are given. It is demonstrated that a rigid area can arise near the friction surface. The behavior of the resultant solution near the friction surfaces is compared with the behavior of known solutions for other models of rigid-plastic materials.

Key words: *singularity, friction, analytical solution, plastic anisotropy.*

In some models of rigid-plastic materials, the solutions are singular near the maximum friction surfaces (in particular, some derivatives of the projections of the velocity vector and the stress tensor components tend to infinity near such surfaces). The general asymptotic analysis of the behavior of solutions was performed in [1] for an arbitrary flow of an ideal rigid-plastic material satisfying an arbitrary smooth yield condition independent of the mean stress, in [2] for an axisymmetric flow of a material satisfying the Tresca yield condition, in [3] for a plane flow, and in [4] for an axisymmetric flow of a material satisfying the double shear model (the model itself is described in [5]). In all cases, the equivalent strain rate (second invariant of the strain rate tensor) near the maximum friction surface during slipping was found to tend to infinity in an inverse proportion to the square root from the distance to this surface. Solutions of problems for various material models show that this property of the velocity field depends on the material model [6–10]. The singular behavior of the velocity field allows new theories to be proposed to describe the changes in the material properties in a thin layer near the friction surfaces [11, 12].

A solution of a model problem of an instantaneous flow of a wedge-shaped layer of a rigid-plastic orthotropic material satisfying the model [13] between rotating plates with the maximum friction law on the plate surfaces is obtained in the present work. Solutions of this problem for different material models make it possible to find some specific features of these solutions, including their singular character [7, 8, 14]. Other solutions of model problems within the framework of the anisotropic plasticity theory were obtained in [15, 16], where the asymptotic analysis of solutions in the vicinity of the maximum friction surfaces was not performed.

Let us consider an instantaneous plane flow of a plastic orthotropic material compressed between two plates rotating with an angular velocity ω with respect to the point O (Fig. 1). Assuming that there is no drain at the point O, we introduce a polar coordinate system (r, θ) with the origin at the point O. As $\theta = 0$ is considered as the axis of symmetry, it is sufficient to obtain the solution at $\theta \geqslant 0$. If the principal axes of anisotropy coincide with the coordinate axes of the polar coordinate system, the yield condition proposed in [13] is written in the form

$$\frac{(\sigma_{rr} - \sigma_{\theta\theta})^2}{4(1-c)} + \sigma_{r\theta}^2 = T^2, \tag{1}$$

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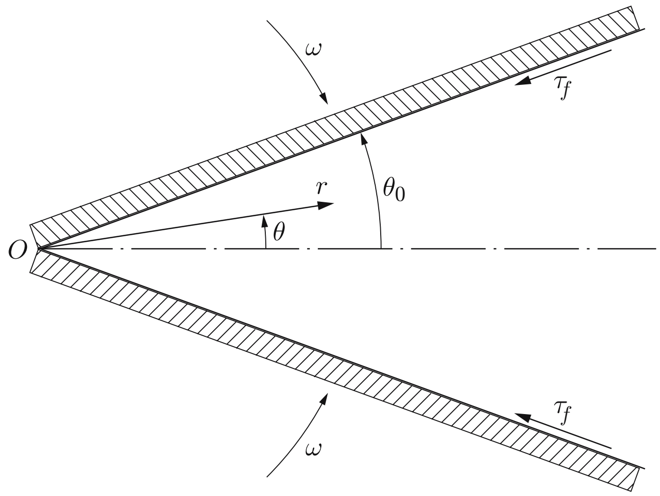

Fig. 1. Geometry of the problem.

where σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are the stress tensor components, T is the shear yield stress in the principal axes of anisotropy, and $1 > c > -\infty$ is a parameter calculated on the basis of the yield stresses of the material in the principal axes of anisotropy [13] (the value $c = 0$ corresponds to an isotropic material). The low rule associated with the yield condition (1) has the form

$$\xi_{rr} = \frac{\lambda}{1-c}\,(\sigma_{rr} - \sigma_{\theta\theta}), \qquad \xi_{\theta\theta} = -\frac{\lambda}{1-c}\,(\sigma_{rr} - \sigma_{\theta\theta}), \qquad \xi_{r\theta} = 2\lambda\sigma_{r\theta}, \tag{2}$$

where ξ_{rr}, $\xi_{\theta\theta}$, and $\xi_{r\theta}$ are the component of the strain rate tensor; $\lambda \geqslant 0$. In the case considered, the equilibrium equations have the form

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{3}$$

and the boundary conditions on the axis of symmetry are

$$\sigma_{r\theta}\Big|_{\theta=0} = 0; \tag{4}$$

$$u_\theta\Big|_{\theta=0} = 0 \tag{5}$$

(u_θ is the circumferential velocity). On the plate surface ($\theta = \theta_0$) (see Fig. 1), we specify the circumferential velocity

$$u_\theta = -\omega r \tag{6}$$

and the friction law. Accepting the maximum friction law and taking into account the direction of the specific friction forces τ_f (see Fig. 1), we obtain the condition

$$\sigma_{r\theta}\Big|_{\theta=\theta_0} = -T, \tag{7}$$

which is valid during slipping; in the case of adhesion, it should be replaced by the condition

$$u_r\Big|_{\theta=\theta_0} = 0, \tag{8}$$

where u_r is the radial velocity. The yield condition (1) is satisfied owing to the substitution

$$\sigma_{rr} = \sigma + T(1-c)^{1/2}\cos 2\psi, \qquad \sigma_{\theta\theta} = \sigma - T(1-c)^{1/2}\cos 2\psi, \qquad \sigma_{r\theta} = -T\sin 2\psi. \tag{9}$$

As for most plastic solutions obtained by the semi-inverse method, we assume that the function ψ depends only on one spatial coordinate, in the case considered, on the coordinate θ. Then, the substitution of relations (9) into the equilibrium equations (3) yields

$$\frac{r}{T}\frac{\partial\sigma}{\partial r} - 2\cos 2\psi\Big(\frac{d\psi}{d\theta} - (1-c)^{1/2}\Big) = 0, \qquad \frac{1}{T}\frac{\partial\sigma}{\partial\theta} + 2\sin 2\psi\Big((1-c)^{1/2}\frac{d\psi}{d\theta} - 1\Big) = 0. \tag{10}$$

It follows from the second equation in (10) that $\partial^2\sigma/\partial r\,\partial\theta = 0$. Then, the first equation in (10) yields

$$\frac{d\psi}{d\theta} = -\frac{A}{\cos 2\psi} + (1-c)^{1/2}, \qquad \frac{\sigma}{T} = -A\ln\frac{r}{R} + \sigma_0(\psi). \tag{11}$$

Here, the constant R is introduced for convenience, and $\sigma_0(\psi)$ is an arbitrary function of ψ. It follows from relations (9), (7) and the natural assumption $\sigma_{rr} > \sigma_{\theta\theta}$ that $0 \leqslant \psi \leqslant \pi/4$; hence, the inequality $d\psi/d\theta > 0$ should be satisfied at $\psi = \pi/4$ (or $\theta = \theta_0$). Then, Eq. (11) yields

$$A \leqslant 0. \tag{12}$$

The solution of the first equation in system (11) can be written in elementary functions, but it is more convenient to write it in the form

$$\theta = \int_0^{\psi} \frac{\cos 2\chi}{(1-c)^{1/2}\cos 2\chi - A}\,d\psi. \tag{13}$$

Solution (13) with allowance for Eq. (9) satisfies the boundary condition (4).

In the case of slipping, it follows from Eqs. (7) and (9) that $\psi = \pi/4$ at $\theta = \theta_0$, and the value of A is found from Eq. (13):

$$\theta_0 = \int_0^{\pi/4} \frac{\cos 2\chi}{(1-c)^{1/2}\cos 2\chi - A}\,d\psi. \tag{14}$$

Taking into account inequality (12) and assuming that $A = 0$, we find, from Eq. (14), the maximum value of θ_0 at which the solution for the case of adhesion is obtained:

$$\theta_{\max} = (1-c)^{-1/2}\pi/4. \tag{15}$$

Substituting the value of σ from the second equation of (11) into the second equation of (10) and eliminating $d\psi/d\theta$ with the help of the first equation of (11), we obtain

$$\frac{d\sigma_0}{d\psi} = 2\sin 2\psi\Big(\frac{c\cos 2\psi + A(1-c)^{1/2}}{(1-c)^{1/2}\cos 2\psi - A}\Big). \tag{16}$$

The solution of Eq. (16) can also be written in elementary functions, but it is more convenient to write it in the form

$$\sigma_0 = 2\int_0^{\psi} \sin 2\chi\Big(\frac{c\cos 2\chi + A(1-c)^{1/2}}{(1-c)^{1/2}\cos 2\chi - A}\Big)\,d\chi + A_0,$$

where A_0 is an arbitrary constant whose value cannot be determined from the boundary conditions because of plastic incompressibility of the material and unboundedness of the layer.

Taking into account Eq. (9), we write Eqs. (2) in the form

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{r\,\partial\theta} + \frac{u_r}{r} = 0, \qquad \Big(\frac{\partial u_r}{\partial r} - \frac{\partial u_\theta}{r\,\partial\theta} - \frac{u_r}{r}\Big)(1-c)^{1/2}\tan 2\psi = -\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{\partial u_r}{r\,\partial\theta}. \tag{17}$$

We use the velocity field in the form

$$u_r = \frac{\omega r}{2}\,\frac{dg(\psi)}{d\theta}, \qquad u_\theta = -\omega r g(\psi), \tag{18}$$

were $g(\psi)$ is an arbitrary function of ψ [the first equation of system (17) is satisfied with an arbitrary choice of this function]. The boundary condition (5) is transformed to

$$g\Big|_{\psi=0} = 0, \tag{19}$$

and the boundary condition (6) in the case of slipping on the friction surface is presented as

$$g\Big|_{\psi=\pi/4} = 1. \tag{20}$$

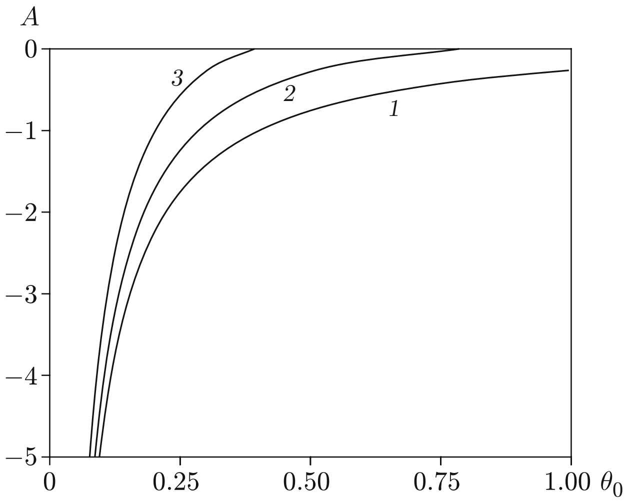

Fig. 2

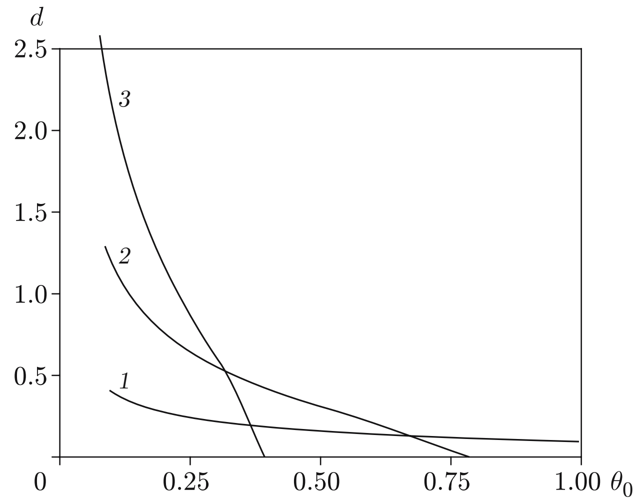

Fig. 3

Fig. 2. Dependence of A on the angle between the plates θ_0 for $c = 0.9$ (1), 0 (isotropic material) (2), and -3 (3).

Fig. 3. Dimensionless strain rate intensity factor d versus the angle between the plates θ_0 for different values of the parameter c (notation the same as in Fig. 2).

Substituting Eq. (18) into the second equation of system (17), assuming that $G = dg/d\theta$, and passing to differentiation with respect to ψ with the use of Eq. (11), we obtain

$$\frac{dG}{d\psi} = \frac{2(1-c)^{1/2} G \sin 2\psi}{A - (1-c)^{1/2}\cos 2\psi}. \tag{21}$$

Integrating Eq. (21), we find $G = G_0[(1-c)^{1/2}\cos 2\psi - A]$ ($G_0 = \text{const}$). Using the solution obtained, the definition for the function G, and relations (11), we obtain the equation $dg/d\psi = G_0 \cos 2\psi$; integration of this equation with allowance for the boundary condition (19) yields

$$g = (1/2) G_0 \sin 2\psi. \tag{22}$$

Taking into account the boundary condition (20), we find $G_0 = 2$. Then, we obtain $g = \sin 2\psi$ from Eq. (22), and Eqs. (18) and (11) yield the velocity field in the form

$$u_r = \omega r[(1-c)^{1/2}\cos 2\psi - A], \qquad u_\theta = -\omega r \sin 2\psi. \tag{23}$$

We determine the equivalent strain rate $\xi_{\rm eq} = \sqrt{2/3}\,(\xi_{rr}^2 + \xi_{\theta\theta}^2 + 2\xi_{r\theta}^2)^{1/2}$ from Eqs. (11) and (23):

$$\xi_{\rm eq} = (2/\sqrt{3}\,)\omega[(1-c)^{1/2}\cos 2\psi - A][1 + (1-c)\tan^2 2\psi]^{1/2}.$$

The equivalent strain rate near the friction surface acquires the form

$$\xi_{\rm eq} = -A(1-c)^{1/2}\omega/[\sqrt{3}\,(\pi/4 - \psi)] + o[(\pi/4 - \psi)^{-1}], \qquad \psi \to \pi/4, \tag{24}$$

and the solution of the first equation of (11) becomes

$$\theta_0 - \theta = -A^{-1}(\pi/4 - \psi)^2 + o[(\pi/4 - \psi)^2], \qquad \psi \to \pi/4. \tag{25}$$

It follows from Eqs. (24) and (25) that

$$\xi_{\rm eq} = \sqrt{-A}\,(1-c)^{1/2}\omega/[\sqrt{3}\,(\theta_0 - \theta)^{1/2}] + o[(\theta_0 - \theta)^{-1/2}], \qquad \theta \to \theta_0. \tag{26}$$

Such an asymptotic presentation of the equivalent strain rate coincides with the corresponding presentation used in some theories of a rigid-plastic body [see, e.g., 1–4, 9, 10]. In particular, Eq. (26) can be used to determine the strain rate intensity factor, which was introduced in [1]:

$$D = \omega[-A(1-c)r/3]^{1/2}. \tag{27}$$

It follows from Eqs. (26) that the singularity in the velocity field disappears at $\theta_0 = \theta_{\max}$ [the quantity $\theta_{\max}$ is determined in Eq. (15) with $A = 0$]. According to Eq, (27), in particular, the strain rate intensity factor vanishes. In addition, it follows from the expression for the radial velocity (23) that the adhesion condition (8) is satisfied in this case.

Thus, the behavior of the solution of the boundary-value problem posed depends on the value of θ_0. According to Eqs. (26), in particular, the equivalent strain rate near the maximum friction surface tends to infinity at $\theta_0 < \theta_{\max}$, and the slipping condition is valid. At $\theta_0 = \theta_{\max}$, the singularity disappears, the material is in the plastic state, and the adhesion condition is satisfied. At $\theta_0 > \theta_{\max}$, the adhesion condition holds, the region $\theta_0 > \theta > \theta_{\max}$ is rigid, and the solution in the region $\theta_{\max} > \theta > 0$ is the same as that in the case with $\theta_0 = \theta_{\max}$.

Figure 2 shows the dependence $A(\theta_0)$ calculated by Eq. (14) for different values of the parameter c characterizing plastic anisotropy. Figure 3 shows the dimensionless strain rate intensity factor $d = D/(\omega r^{1/2})$ as a function of θ_0, which was calculated by Eq. (27), for different values of the parameter c. It is seen from Fig. 3 that plastic anisotropy exerts a significant effect on the magnitude of the strain rate intensity factor. It should be taken into account in calculations based on the theories [11, 12].

Thus, the solution near the friction surfaces is demonstrated to be singular in the general case, and conditions are given at which the singularity disappears.

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